Polynomials having no Zero in a Given Region

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ABSTRACT:

In this paper we consider some polynomials having no zeros in a given region. Our results when combined with some known results give ring –shaped regions containing a specific number of zeros of the polynomial.

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I. INTRODUCTION AND STATEMENT OF RESULTS

In the literature we find a large number of published research papers concerning the number of zeros of a polynomial in a given circle. For the class of polynomials with real coefficients, Q. G. Mohammad [5] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

Bidkham an d Dewan [1] generalized Theorem A in the following way:

Theorem B: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \le a_{n-1} \le ...... \le a_{k+1} \le a_k \ge a_{k-1} \ge ...... \ge a_1 \ge a_0 > 0,$$

for some $k, 0 \le k \le n$ .Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{|a_n|+|a_0|-a_n-a_0+2a_k}{|a_0|}\right\}.$$

Ebadian et al [2] generalized the above results by proving the following results:

Theorem C: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \le a_{n-1} \le ...... \le a_{k+1} \le a_k \ge a_{k-1} \ge ...... \ge a_0$$

for some k, $0 \le k \le n$ .Then the number of zeros of P(z) in $|z| \le \frac{R}{2}$,R>0, does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{|a_n|R^{n+1}+|a_0|+R^k(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\} \text{ for } R \ge 1$$

and

$$\frac{1}{\log 2}\log\left\{\frac{|a_n|R^{n+1}+|a_0|+R(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\} \text{ for } R\le 1 .$$

M.H.Gulzar [3] generalized the above result by proving the following result:

Theorem D: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ such that for some $k,\tau,\lambda, 0<k\le 1, 0<\tau\le 1,\ 0\le\lambda\le n$,

$$k\alpha_n\le\alpha_{n-1}\le......\le\alpha_{\lambda+1}\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge......\ge\tau\alpha_0 .$$

Then the number of zeros of P(z) in $|z|\le\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^{\lambda}[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\ +R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\right\}$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\ +R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\right\}$$

In this paper we prove the following result:

Theorem 1: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ such that for some $k,\tau,\lambda, 0<k\le 1, 0<\tau\le 1,\ 0\le\lambda\le n$,

$$k\alpha_n\le\alpha_{n-1}\le......\le\alpha_{\lambda+1}\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge......\ge\tau\alpha_0 .$$

Then P(z) has no zero in $|z|<\frac{|a_0|}{M_1}$ for $R\ge 1$ and no zero in $|z|<\frac{|a_0|}{M_2}$ for $R\le 1$

where

$$M_1=|a_n|R^{n+1}+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n}|\beta_j|]$$
$$+R^{\lambda}[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]$$

and

$$M_2=|a_n|R^{n+1}+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n}|\beta_j|]$$
$$+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|].$$

Combining Theorem 1 with Theorem D, we get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ such that for some $k,\tau,\lambda, 0<k\le 1, 0<\tau\le 1,\ 0\le\lambda\le n$,

$$k\alpha_n\le\alpha_{n-1}\le......\le\alpha_{\lambda+1}\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge......\ge\tau\alpha_0 .$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M_1}\le|z|\le\frac{R}{c}\,(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\begin{array}{l}\dfrac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^{\lambda}[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\end{array}\right\}$$

for $R\ge 1$

and the number of zeros of P(z) in $\frac{|a_0|}{M_2}\le|z|\le\frac{R}{c}\,(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\begin{array}{l}\dfrac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\end{array}\right\}$$

for $R\le 1$,

where M_1 and M_2 are as given in Theorem 1.

For different values of the parameters, we get many interesting results including some already known results.

2. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

F(z) =(1-z)P(z)

$$=(1-z)(a_n z^n+a_{n-1}z^{n-1}+......+a_1 z+a_0)$$

$$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+......+(a_1-a_0)z+a_0$$

$$=-a_n z^{n+1}+a_0+[(k\alpha_n-\alpha_{n-1})-(k-1)\alpha_n]z^n+\sum_{j=\lambda+1}^{n-1}(\alpha_j-\alpha_{j-1})z^j$$

$$+\sum_{j=2}^{\lambda}(\alpha_j-\alpha_{j-1})z^j+[(\alpha_1-\tau\alpha_0)+(\tau-1)\alpha_0]z+i\{\sum_{j=\lambda+1}^{n}(\beta_j-\beta_{j-1})z^j$$

$$+\sum_{j=1}^{\lambda}(\beta_j-\beta_{j-1})z^j\}$$

$=a_0+G(z)$, where

$$G(z)=-a_n z^{n+1}+a_0+[(k\alpha_n-\alpha_{n-1})-(k-1)\alpha_n]z^n+\sum_{j=\lambda+1}^{n-1}(\alpha_j-\alpha_{j-1})z^j$$
$$+\sum_{j=2}^{\lambda}(\alpha_j-\alpha_{j-1})z^j+[(\alpha_1-\tau\alpha_0)+(\tau-1)\alpha_0]z+i\sum_{j=1}^{\lambda}(\beta_j-\beta_{j-1})z^j$$

For $|z|\le R$, we have, by using the hypothesis

$$|G(z)|\le|a_n|R^{n+1}+[(\alpha_{n-1}-k\alpha_n)+(1-k)|\alpha_n|]R^n+\sum_{j=\lambda+1}^{n-1}(\alpha_{j-1}-\alpha_j)R^j$$
$$+\sum_{j=1}^{\lambda}(\alpha_j-\alpha_{j-1})R^j+[(\alpha_1-\tau\alpha_0)+(1-\tau)|\alpha_0|]R+\sum_{j=1}^{n}(|\beta_j|+|\beta_{j-1}|)R^j$$

which gives

$$|G(z)|\le|a_n|R^{n+1}+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda]+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n}|\beta_j|]$$
$$+R^\lambda[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]$$
$$=M_1 \quad \text{for } R\ge 1$$

and

$$|G(z)|\le|a_n|R^{n+1}+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda]+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n}|\beta_j|]$$
$$+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]$$
$$=M_2 \quad \text{for } R\le 1.$$

Since G(z) is analytic in $|z|\le R$ and G(0)=0, it follows by Schwarz Lemma that

$|G(z)|\le M_1|z|$ for $R\ge 1$ and $|G(z)|\le M_2|z|$ for $R\le 1$.

Hence, for $R\ge 1$,

$$\begin{aligned}|F(z)|&=|a_0+G(z)|\\&\ge|a_0|-|G(z)|\\&\ge|a_0|-M_1|z|\\&>0\end{aligned}$$

if

$$|z|<\frac{|a_0|}{M_1}.$$

And for $R\le 1$,

$$\begin{aligned}|F(z)|&=|a_0+G(z)|\\&\ge|a_0|-|G(z)|\\&\ge|a_0|-M_2|z|\\&>0\end{aligned}$$

if

$$|z|<\frac{|a_0|}{M_2}.$$

This shows that F(z) has no zero in $|z|<\dfrac{|a_0|}{M_1}$ for $R\ge 1$ and no zero in $|z|<\dfrac{|a_0|}{M_2}$ for $R\le 1$. But the zeros of P(z) are also the zeros of F(z). Therefore, the result follows.

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